Unanticipated Shocks and Forward Guidance at the ZLB

Appendix

Not For Publication

Callum Jones

1 Smets and Wouters (2007) model

The linearized equations of the Smets and Wouters (2007) model are the following equations, plus the monetary policy rule. Expectations are dropped for brevity.

1.1 Sticky price economy

Factor prices:

$$mc_t = \alpha r_t + (1-\alpha) w_t - \epsilon_{a,t} \tag{1}$$

$$r_t = w_t + l_t - k_t^s \tag{2}$$

$$z_t = \tfrac{1-\psi}{\psi} r_t \tag{3}$$

Investment:

$$i_t = \tfrac{1}{1+\bar{\beta}\gamma}\left(i_{t-1} + \bar{\beta}\gamma i_{t+1} + \tfrac{1}{\gamma^2 \phi} q_t\right) + \epsilon_{i,t} \tag{4}$$

$$q_t = \tfrac{\sigma_c(1+\lambda/\gamma)}{1-\lambda/\gamma}\epsilon_{b,t} + \tfrac{1-\delta}{1-\delta+R^k}\, q_{t+1} + \tfrac{R^k}{1-\delta+R^k}\, r_{t+1} - r_t + \pi_{t+1} \tag{5}$$

Consumption decision:

$$c_t = \epsilon_{b,t} + \tfrac{\lambda/\gamma}{1+\lambda/\gamma}\, c_{t-1} + \tfrac{1}{1+\lambda/\gamma} c_{t+1} + \tfrac{(\sigma_c - 1)\, W^* L^*/C^*}{\sigma_c,(1+\lambda/\gamma)}\, \left(l_t - l_{t+1}\right) - \tfrac{1-\lambda/\gamma}{\sigma_c,(1+\lambda/\gamma)}\, \left(r_t - \pi_{t+1}\right) \tag{6}$$

Resource constraint:

$$y_t = c_t c_y + i_t i_y + \epsilon_{g,t} + z_t z_y \tag{7}$$

Production function:

$$y_t = \phi_p\left(\epsilon_{a,t} + \alpha k_t^s + (1-\alpha)\; l_t\right) \tag{8}$$

$$k_t^s = z_t + k_{t-1} \tag{9}$$

Evolution of capital:

$$k_t = (1 - i_k)\; k_{t-1} + i_k i_t + \epsilon_{i,t}\, \phi\, \gamma^2\, i_k \tag{10}$$

Price and wage Philips curves:

$$\pi_t = \tfrac{1}{1+\bar{\beta}\gamma\, \iota_p}\left(\bar{\beta}\gamma\, \pi_{t+1} + \iota_p\, \pi_{t-1} + mc_t\, \frac{\frac{(1-\xi_p)\,(1-\bar{\beta}\gamma\,\xi_p)}{\xi_p}}{1+(\phi_p - 1)\,\epsilon_p}\right) + \epsilon_{p,t} \tag{11}$$

$$w_t = w_1 w_{t-1} + w_2 w_{t+1} + w_3 \pi_{t-1} - w_4 \pi_t + w_2 \pi_{t+1} + w_5\left(\sigma_l\, l_t + \tfrac{1}{1-\lambda/\gamma}\, c_t - \tfrac{\lambda/\gamma}{1-\lambda/\gamma}\, c_{t-1} - w_t\right) + \epsilon_{w,t} \tag{12}$$

where $w_1 = \frac{1}{1+\bar{\beta}\gamma}$, $w_2 = \frac{\bar{\beta}\gamma}{1+\bar{\beta}\gamma}$, $w_3 = \frac{\iota_w}{1+\bar{\beta}\gamma}$, $w_4 = \frac{1+\bar{\beta}\gamma\,\iota_w}{1+\bar{\beta}\gamma}$, and $w_5 = \frac{(1-\xi_w)\left(1-\bar{\beta}\gamma\,\xi_w\right)}{\left(1+\bar{\beta}\gamma\right)\xi_w}\, \frac{1}{1+(\phi_w - 1)\,\epsilon_w}$.

1.2 Flexible price economy

The corresponding equations defining the flexible price economy are:

$$\epsilon_{a,t} = \alpha r_t^f + (1-\alpha) w_t^f \tag{13}$$

$$r_t^f = w_t^f + l_t^f - k_t^f \tag{14}$$

$$z_t^f = \tfrac{1-\psi}{\psi} r_t^f \tag{15}$$

$$k_t^f = z_t^f + kp_{t-1}^f \tag{16}$$

$$i_t^f = \tfrac{1}{1+\bar\beta\gamma} \left(i_{t-1}^f + \bar\beta\gamma\, i_{t+1}^f + \tfrac{1}{\gamma^2\,\phi}\, q_t^f \right) + \varepsilon_{i,t} \tag{17}$$

$$q_t^f = \tfrac{1-\delta}{1-\delta+Rk}\, q_{t+1}^f + \tfrac{R^k}{1-\delta+R^k}\, rk_{t+1}^f - rr_t^f + \tfrac{\sigma_c,(1+\lambda/\gamma)}{1-\lambda/\gamma}\, \epsilon_{b,t} \tag{18}$$

$$c_t^f = \epsilon_{b,t} + \tfrac{\lambda/\gamma}{1+\lambda/\gamma}\, c_{t-1}^f + \tfrac{1}{1+\lambda/\gamma}\, c_{t+1}^f + \tfrac{(\sigma_c-1)\, W^* L^*/C^*}{\sigma_c,(1+\lambda/\gamma)} \left(l_t^f - l_{t+1}^f \right) - \tfrac{1-\lambda/\gamma}{\sigma_c,(1+\lambda/\gamma)} rr_t^f \tag{19}$$

$$y_t^f = c_t^f\, c_y + i_t^f\, i_y + \epsilon_{g,t} + z_t^f\, z_y \tag{20}$$

$$y_t^f = \phi_p \left(\epsilon_{a,t} + \alpha k_t^f + (1-\alpha)\ l_t^f \right) \tag{21}$$

$$k_t^{p,f} = k_{t-1}^{p,f}\ (1 - i_k) + i_t^f\ i_k + \epsilon_{i,t}\, \gamma^2\, \phi\, i_k \tag{22}$$

$$w_t^f = \sigma_l l_t^f + \tfrac{1}{1-\lambda/\gamma} c_t^f - \tfrac{\lambda/\gamma}{1-\lambda/\gamma} c_{t-1}^f \tag{23}$$

1.3 Shocks

$$\epsilon_{a,t} = \rho_a\, \epsilon_{a,t-1} + \sigma_a\, \eta_{a,t} \tag{24}$$

$$\epsilon_{b,t} = \rho_b\, \epsilon_{b,t-1} + \sigma_b\, \eta_{b,t} \tag{25}$$

$$\epsilon_{g,t} = \rho_g\, \epsilon_{g,t-1} + \sigma_g\, \eta_{g,t} + \eta_{a,t}\, \sigma_a\, \rho_{ga} \tag{26}$$

$$\epsilon_{i,t} = \rho_i\, \epsilon_{i,t-1} + \sigma_i\, \eta_{i,t} \tag{27}$$

$$\epsilon_{r,t} = \rho_r\, \epsilon_{r,t-1} + \sigma_r\, \eta_{r,t} \tag{28}$$

$$\epsilon_{p,t} = \rho_p\, \epsilon_{p,t-1} + \eta_{p,ma,t} - \mu_p\, \eta_{p,ma,t-1} \tag{29}$$

$$\eta_{p,ma,t} = \sigma_p\, \eta_{p,t} \tag{30}$$

$$\epsilon_{w,t} = \rho_w\, \epsilon_{w,t-1} + \eta_{w,ma,t} - \mu_w\, \eta_{w,ma,t-1} \tag{31}$$

$$\eta_{w,ma,t} = \sigma_w\, \eta_{w,t} \tag{32}$$

1.4 Measurement equations

$$dy_t = \bar\gamma + y_t - y_{t-1} \tag{33}$$

$$dc_t = \bar\gamma + c_t - c_{t-1} \tag{34}$$

$$di_t = \bar\gamma + i_t - i_{t-1} \tag{35}$$

$$dw_t = \bar\gamma + w_t - w_{t-1} \tag{36}$$

$$\pi_t^{obs} = \bar\pi + \pi_t \tag{37}$$

$$r_t^{obs} = \bar{r} + r_t \tag{38}$$

$$l_t^{obs} = \bar{l} + l_t \tag{39}$$

2 Solution methods

2.1 Binder and Pesaran (1995)

Consider a rational expectations model $x_t = \Psi(x_{t-1}, \mathbb{E}_t x_{t+1}, w_t)$. Linearize the model around a non-stochastic steady state to get:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}w_t.$$

In an economy where all agents know the regime and expectations are formed under that regime, the solution is a reduced-form VAR:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}w_t.$$

where $\mathbf{J}$, $\mathbf{Q}$ and $\mathbf{G}$ are conformable matrices which are functions of the structural matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{F}$. As in Binder and Pesaran (1995) and Kulish and Pagan (2017), $\mathbf{Q}$ is solved by iterating on the quadratic expression:

$$\mathbf{Q} = [\mathbf{A} - \mathbf{D}\mathbf{Q}]^{-1}\mathbf{B}.$$

With $\mathbf{Q}$ in hand, we compute $\mathbf{J}$ and $\mathbf{G}$ with:

$$\mathbf{J} = [\mathbf{A} - \mathbf{D}\mathbf{Q}]^{-1}(\mathbf{C} + \mathbf{D}\mathbf{J})$$
$$\mathbf{G} = [\mathbf{A} - \mathbf{D}\mathbf{Q}]^{-1}\mathbf{F}.$$

2.2 Sims (2002)

The following is adapted from Cagliarini and Kulish (2013). Write a model in matrix form as:

$$\widetilde{\Gamma}_0\mathbf{y}_t = \widetilde{\Gamma}_1\mathbf{y}_{t-1} + \widetilde{C} + \widetilde{\Psi}\varepsilon_t, \tag{40}$$

where the state vector is defined by and ordered according to:

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{y}_{1,t} \\ \mathbf{y}_{2,t} \\ \mathbb{E}_t\mathbf{z}_{t+1} \end{bmatrix},$$

and where $\mathbf{y}_{1,t}$ is an $(n_1 \times 1)$ vector of exogenous and some endogenous variables, and $\mathbf{y}_{2,t}$ is an $(n_2 \times 1)$ vector with those endogenous variables for which conditional expectations appear; $\mathbf{z}_{t+1}$, $(k \times 1)$, where $k = n_2 \times s$ contains s leads of $\mathbf{y}_{2,t}$; in most models like Ireland (2004) and Smets and Wouters (2007), however, $\mathbf{z}_{t+1} = \mathbf{y}_{2,t+1}$ so that $s = 1$ and $k = n_2$. The dimension of $\mathbf{y}_t$ is $n \times 1$, where $n = n_1 + n_2 + k$. Also, we assume ε_t to be an $l \times 1$ vector of serially uncorrelated processes, $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are $(n_1 + n_2) \times n$ matrices, $\widetilde{C}$ is $(n_1 + n_2) \times 1$ and $\widetilde{\Psi}$ is $(n_1 + n_2) \times l$.

Because of the presence of expectations, we cannot invert $\widetilde{\Gamma}$ and estimate a reduced form version of (40). Sims's (2002) proposal is to append to (40) expectations revisions which will be solved as part of the solution. Let η_t be the vector of expectations revisions:

$$\eta_t = \mathbb{E}_t \mathbf{z}_t - \mathbb{E}_{t-1} \mathbf{z}_t, \tag{41}$$

where $\mathbb{E}_t \eta_{t+j} = 0$ for $j \geq 1$. For example, if $z_t = y_{2,t}$, then η_t are forecast revisions.

Augment the system defined by Equation (40) with the k equations from Equation (41) to obtain:

$$\Gamma_0 \mathbf{y}_t = C + \Gamma_1 \mathbf{y}_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{42}$$

where the matrices Γ_0, Γ_1,C,Ψ, and Π are of conformable dimensions. Γ_0 is now an $n \times n$ matrix, which we will invert with a Schur (QZ) decomposition, and impose conditions such that we can remove the η_t from the system.

To solve (42) as Sims (2002), take a Schur (QZ) decomposition of (Γ_0, Γ_1) to get:

$$Q' \Lambda Z' = \Gamma_0 \quad \text{and} \quad Q' \Omega Z' = \Gamma_1,$$

where Λ and Ω are both upper triangular. The matrices Q and Z are unitary, so that $QQ' = I$ and $ZZ' = I$. Pre-multiply model equation by Q and define $w_t = Z' y_t$ to rewrite the system as:

$$\Lambda w_t = \Omega w_{t-1} + Q(C + \Psi \varepsilon_t + \Pi \eta_t).$$

Define $w_{1,t} = Z_1' y_t$ and $w_{2,t} = Z_2' y_t$. Λ and Ω are upper triangular and have the property that the generalized eigenvalues of (Γ_0, Γ_1) are ratios of diagonal elements of Ω and Λ. Rearrange the system so that the explosive eigenvalues correspond to the lower right blocks of Λ and Σ, partitioning w_t and rewriting the system as:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} (C + \Psi \varepsilon_t + \Pi \eta_t).$$

The lower block of the system are those equations which correspond to the m explosive generalised eigenvalues of (Γ_0, Γ_1). The lower set of equations are not affected by $w_{1,t}$. Isolate these:

$$\Lambda_{22} w_{2,t} = \Omega_{22} w_{2,t-1} + Q_2 \left(C + \Psi \varepsilon_t + \Pi \eta_t \right).$$

For stability of the system, we need η_t to offset the effect of ε_t on $w_{2,t}$. To see this, first solve $w_{2,t}$ forward to get:

$$w_{2,t} = \left(\Lambda_{22} - \Omega_{22} \right)^{-1} Q_2 C - \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22} \right)^{j-1} \Omega_{22}^{-1} Q_2 (\Psi \varepsilon_{t+j} + \Pi \eta_{t+j}).$$

This says that $w_{2,t}$ requires having in hand all future values of ε_t and η_t at time t. Take expectations of this expression at time t to get:

$$w_{2,t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 C - \mathbb{E}_t \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j}.$$

Also take expectations at time $t+1$ to get:

$$w_{2,t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 C - \mathbb{E}_{t+1} \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j} - \Omega_{22}^{-1} Q_2 \Pi \eta_{t+1}.$$

Note the left hand side has not changed, so equating these two expressions implies:

$$Q_2 \Pi \eta_{t+1} = \Omega_{22} \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 \left(\mathbb{E}_t \epsilon_{t+j} - \mathbb{E}_{t+1} \epsilon_{t+j}\right).$$

This says that for the system to be stable, the expectations revisions must offset the effect that shocks ε_t have on the explosive component of the system, $w_{2,t}$. Expectations revisions ensure that the system is placed on the saddle path to stability. For this to be true, Sims (2002) shows that what is required for a unique solution is that the number of explosive eigenvalues of (Γ_0, Γ_1), m equals the number of variables which appear as expectations in the system, k. Under this condition, the system is on a saddle path to a steady-state from any initial condition. (Note, there are weaker conditions just for stability.) If this is true, and if the solution is stable then there is a matrix Φ such that:

$$Q_1 \Pi = \Phi Q_2 \Pi.$$

By premultiplying the system by $[I_{n-p}, -\Phi]$, the coefficient on η_t is:

$$Q_1 \Pi - \Phi Q_2 \Pi.$$

Since existence of a solution requires $Q_1 \Pi = \Phi Q_2 \Pi$, the η_t drop out of (40), so that a solution to the model can be written as:

$$\mathbf{y}_t = S_0 + S_1 \mathbf{y}_{t-1} + S_2 \varepsilon_t + S_y \mathbb{E}_t \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j}, \tag{43}$$

where:

$$H = Z \begin{bmatrix} \Sigma_{11}^{-1} & -\Sigma_{11}^{-1} (\Sigma_{12} - \Phi \Sigma_{22}) \\ 0 & I \end{bmatrix}, \qquad S_0 = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ (\Sigma_{22} - \Omega_{22})^{-1} Q_2 \end{bmatrix} C,$$

$$S_1 = H \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi \Omega_{22} \\ 0 & 0 \end{bmatrix}, \qquad S_2 = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ 0 \end{bmatrix} \Psi,$$

$$S_y = -H \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

The solution (43) is in the desired VAR(1) form.

2.3 Foreseen structural changes

Suppose the structural parameters of the economy are known to change into the future. In particular, suppose the economy is expected to evolve with the following structure: in time period $t = 1$, the economy starts with the following structure:

$$\widetilde{\Gamma}_{0,1}\mathbf{y}_t = \widetilde{\Gamma}_{1,1}\mathbf{y}_0 + \widetilde{C}_1 + \widetilde{\Psi}_1\varepsilon_1,$$

and in time periods $2 \leq t \leq T$, the structural parameters of the economy evolve according to:

$$\Gamma_{0,t}\mathbf{y}_t = \Gamma_{1,t}\mathbf{y}_{t-1} + C_t + \Pi\eta_t + \Psi_t\left(\varepsilon_t^u + \varepsilon_t^a\right),$$

where ε_t^u are shocks which are unanticipated at time period $t = 1$, ε_t^a are shocks which are anticipated at $t = 1$. In this implementation, expectations revisions are included as the system evolves. Unanticipated shocks are added to show that it is possible to solve the model subject to foreseen structural changes and unanticipated shocks, though the solution would need to be computed each time period. Also notice that the matrices specifying the structural parameters are time-varying. After time period $T+1$, the structural parameters of the economy are fixed, so that the system becomes:

$$\bar{\Gamma}_0\mathbf{y}_t = \bar{\Gamma}_1\mathbf{y}_{t-1} + \bar{C} + \bar{\Pi}\eta_t + \bar{\Psi}\varepsilon_t.$$

Stacking $T \times (n_1 + n_2 + k) + \widetilde{m} - k$ equations and imposing $\mathbb{E}_1\eta_2 = \mathbb{E}_1\eta_3 = \ldots = \mathbb{E}_1\eta_T = 0$ (rational expectations) yields:

$$\begin{bmatrix} \widetilde{\Gamma}_{0,1} & 0 & \ldots & \ldots & 0 \\ -\Gamma_{1,2} & \Gamma_{0,2} & \ddots & & \vdots \\ 0 & -\Gamma_{1,3} & \Gamma_{0,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -\Gamma_{1,T} & \Gamma_{0,T} \\ 0 & \ldots & \ldots & 0 & \bar{Z}_2' \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbb{E}_1\mathbf{y}_2 \\ \vdots \\ \mathbb{E}_1\mathbf{y}_T \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 + \widetilde{\Gamma}_{1,1}\mathbf{y}_0 + \widetilde{\Psi}_1\varepsilon_1 \\ C_2 + \Psi_2\varepsilon_2^a \\ \vdots \\ C_T + \Psi_T\varepsilon_T^a \\ \widetilde{w}_{2,T} \end{bmatrix}. \tag{44}$$

More concisely:

$$A\mathcal{Y} = \mathbf{b}.$$

The necessary condition to invert A is that the final (bar) structure of the economy has a solution, which ensures the economy reaches its saddle path. In particular, Cagliarini and Kulish (2013) show that uniqueness of final system is necessary for the intermediate path of the economy to be unique.

Practically, $\mathcal{Y} = (\mathbf{y}_1', \ldots, \mathbf{y}_T')$ is a $T \times (n_1 + n_2 + k)$ vector. As discussed in Sims (2002) and in Section 2.2, a solution to the final system implies $\bar{m} = k$. This condition implies A is a square matrix. Given uniqueness of the solution to the final system, it is necessary for A be full rank for the path $\mathcal{Y}$ to be unique. This is generally the case unless perverse parameters are used. The system can largely be unconstrained in the intermediate stage. If the system is on a saddle path eventually, there is usually a unique path.

3 Kalman filter and sampler

3.1 Kalman filter

The model in state space representation is:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t w_t \tag{45}$$
$$z_t = \mathbf{H}_t x_t. \tag{46}$$

The error is distributed $w_t \sim N(0, \mathbf{Q})$ where $\mathbf{Q}$ is the covariance matrix of w_t. By assumption, there is no observation error of the data given in the vector z_t. The Kalman filter recursion is given by the following equations, conceptualized as the predict and update steps. The state of the system is $(\hat{x}_t, \mathbf{P}_{t-1})$. In the predict step, the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ are used to compute a forecast of the state $\hat{x}_{t|t-1}$ and the forecast covariance matrix $\mathbf{P}_{t|t-1}$ as:

$$\begin{aligned}\hat{x}_{t|t-1} &= \mathbf{J}_t + \mathbf{Q}_t \hat{x}_t \\ \mathbf{P}_{t|t-1} &= \mathbf{Q}_t \mathbf{P}_{t-1} \mathbf{Q}_{t|t-1}^\top + \mathbf{G}_t \mathbf{Q} \mathbf{G}_t^\top.\end{aligned}$$

This formulation differs from the time-invariant Kalman filter because in the forecast stage the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ can vary over time. We update these forecasts with imperfect observations of the state vector. This update step involves computing forecast errors $\widetilde{y}_t$ and its associated covariance matrix $\mathbf{S}_t$ as:

$$\begin{aligned}\widetilde{y}_t &= z_t - \mathbf{H}_t \hat{x}_{t|t-1} \\ \mathbf{S}_t &= \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^\top.\end{aligned}$$

The Kalman gain matrix is given by:

$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^\top \mathbf{S}_t^{-1}.$$

With $\widetilde{y}_t$, $\mathbf{S}_t$ and $\mathbf{K}_t$ in hand, the optimal filtered update of the state x_t is

$$\hat{x}_t = \hat{x}_{t|t-1} + \mathbf{K}_t \widetilde{y}_t,$$

and for its associated covariance matrix:

$$\mathbf{P}_t = (I - \mathbf{K}_t \mathbf{H}_t) \, \mathbf{P}_{t|t-1}.$$

The Kalman filter is initialized with x_0 and $\mathbf{P}_0$ determined from their unconditional moments, and is computed until the final time period T of data.

3.2 Kalman smoother

With the estimates of the parameters and durations in hand at time period T, the Kalman smoother gives an estimate of $x_{t|T}$, or an estimate of the state vector at each point in time given all available information (see Hamilton, 1994). With $\hat{x}_{t|t-1}$, $\mathbf{P}_{t|t-1}$, $\mathbf{K}_t$ and $\mathbf{S}_t$ in hand from the Kalman filter, the vector $x_{t|T}$ is:

$$x_{t|T} = \hat{x}_{t|t-1} + \mathbf{P}_{t|t-1} r_{t|T},$$

where the vector $r_{T+1|T} = 0$ and is updated with the recursion:

$$r_{t|T} = \mathbf{H}_t^\top \mathbf{S}_t^{-1} \left(z_t - \mathbf{H}_t \hat{x}_{t|t-1} \right) + \left(I - \mathbf{K}_t \mathbf{H}_t \right)^\top \mathbf{P}_{t|t-1}^\top r_{t+1|T}.$$

Finally, to get an estimate of the shocks to each state variable, denoted by e_t, we compute:

$$e_t = \mathbf{G}_t w_t = \mathbf{G}_t r_{t|T}.$$

From these, we get an estimate of the structural shocks used to compute counterfactuals with the model.

3.3 Sampler

This section describes the sampler used to construct an estimate of the posterior distributions. Denote by ϑ the vector of parameters to be estimated, $\mathbf{U}$ the vector of breaking structural parameters (trend growth), and $\mathbf{T}$ the vector of durations to be estimated. Denote by $\mathbf{z} = \{z_\tau\}_{\tau=1}^{T}$ the sequence of observable vectors. The posterior $\mathcal{P}(\vartheta, \mathbf{U}, \mathbf{T} \mid \mathbf{z})$ satisfies:

$$\mathcal{P}(\vartheta, \mathbf{U}, \mathbf{T} \mid \mathbf{z}) \propto \mathcal{L}(\mathbf{z} \mid \vartheta, \mathbf{U}, \mathbf{T}) \times \mathcal{P}(\vartheta, \mathbf{U}, \mathbf{T}).$$

With Gaussian errors, the likelihood function $\mathcal{L}(\mathbf{z} \mid \vartheta, \mathbf{U}, \mathbf{T})$ is computed using the appropriate sequence of structural matrices and the Kalman filter:

$$\log \mathcal{L}(\mathbf{z} \mid \vartheta, \mathbf{U}, \mathbf{T}) = -\left(\frac{N_z T}{2} \right) \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log \det \mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top - \frac{1}{2} \sum_{t=1}^{T} \widetilde{y}_t^\top \left(\mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top \right)^{-1} \widetilde{y}_t.$$

The prior is simply computed using priors over ϑ which are consistent with the literature, and with flat priors on $\mathbf{U}$ and $\mathbf{T}$.[1]

The Markov Chain Monte Carlo posterior sampler has three blocks, corresponding to ϑ, $\mathbf{U}$ and $\mathbf{T}$. Initialize the sampler at step j with the last accepted draw of the structural parameters, the period of the breaking parameters and durations, denoted by ϑ_{j-1}, $\mathbf{U}_{j-1}$ and $\mathbf{T}_{j-1}$ respectively. The three blocks are, in order of computation:

1. In the first block, propose a new $\mathbf{U}_j$ by randomly choosing a date between $[T_1, T_2]$, where T_1 and T_2 are the bounds of the interquartile range of the sample. With $\mathbf{U}_j$, recompute the sequence of structural matrices associated with $(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1})$, compute the posterior $\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1} \mid \mathbf{z})$, and accept $(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1})$ with probability $\frac{\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1} \mid \mathbf{z})}{\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_{j-1} \mid \mathbf{z})}$. If $(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1})$ is accepted, then set $\mathbf{U}_{j-1} = \mathbf{U}_j$.

2. In the second block, randomly choose up to $\bar{T}$ durations to test, corresponding to up to $\bar{T}$ time periods that the economy is at the ZLB. For each of those time periods,

[1] I require the structural break date to lie in the interquartile range of the sample to avoid issues of erroneous errors found in short samples, and I require that each estimated duration lies below some maximum value T^* which, in practice, is rarely visited by the sampler.

randomly choose a duration in the interval $[1, T^*]$ and mix that value with the previously accepted draw to generate a new $\mathbf{T}_j$ proposal. As with the first block, recompute the sequence of structural matrices associated with $(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j)$, compute the posterior $\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1} \mid \mathbf{z})$, and accept the proposal $(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j)$ with probability $\frac{\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j | \mathbf{z})}{\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_{j-1} | \mathbf{z})}$. If $(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j)$ is accepted, then set $\mathbf{T}_{j-1} = \mathbf{T}_j$.

3. The third block is a more standard Metropolis-Hastings random walk step. First start by selecting which structural parameters to propose a new value for. For those parameters, draw a new proposal ϑ_j from a proposal density centered at ϑ_{j-1} and with thick tails to ensure sufficient coverage of the parameter space and an acceptance rate of roughly 20%. The proposal ϑ_j is accepted with probability $\frac{\mathcal{P}(\vartheta_j, \mathbf{U}_{j-1}, \mathbf{T}_{j-1} | \mathbf{z})}{\mathcal{P}(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_{j-1} | \mathbf{z})}$. If $(\vartheta_j, \mathbf{U}_{j-1}, \mathbf{T}_{j-1})$ is accepted, then set $\vartheta_{j-1} = \vartheta_j$.

3.4 Data

I use the same data sources and construction as Smets and Wouters (2007). For wages, I use the nonfarm business sector real compensation per hour, with code COMPRNFB on the Federal Reserve Economic Database.

4 A worked example of the algorithm

Consider the simple example, log-linearized around steady-state where y_t is output and the nominal interest rate i_t ignores the ZLB:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t - \bar{i} &= \rho\,(i_{t-1} - \bar{i}) + \gamma y_t. \end{aligned}$$

Putting this model in the form of (40) requires $\mathbf{y}_t = [\ i_t \quad y_t \quad \mathbb{E}_t y_{t+1}\ ]'$ and:

$$\widetilde{\Gamma}_0 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -\gamma & 0 \end{bmatrix}, \quad \widetilde{\Gamma}_1 = \begin{bmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} \bar{i} \\ \bar{i}(1-\rho) \end{bmatrix}, \quad \widetilde{\Psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Adding expectations revisions requires appending a single equation:

$$\eta_t = y_t - \mathbb{E}_{t-1} y_t.$$

The matrices become:

$$\Gamma_0 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -\gamma & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} \bar{i} \\ \bar{i}(1-\rho) \\ 0 \end{bmatrix}, \quad \widetilde{\Psi} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $\Pi = [\ 0 \quad 0 \quad 1\ ]'$. The routines of Sims (2002) are used to obtain the linear system:

$$\mathbf{y}_t = S_0 + S_1 \mathbf{y}_{t-1} + S_2 \varepsilon_t. \tag{47}$$

The algorithm proceeds as follows. Given a shock ε_t:

1. Using the reduced form system without the ZLB (47), obtain the path $\mathbf{y}_t$ up to some large T. Assume no future shocks:

$$\begin{aligned}\mathbf{y}_t &= S_0 + S_1\mathbf{y}_{t-1} + S_2\varepsilon_t\\ \mathbf{y}_{t+1} &= S_0 + S_1\mathbf{y}_t\\ &\vdots\\ \mathbf{y}_T &= S_0 + S_1\mathbf{y}_{T-1}.\end{aligned}$$

2. Examine $\{i_\tau\}_{\tau=t}^T$. If $i_\tau > 0\ \forall\ \tau$, then stop the algorithm. Otherwise, move to the next step.

3. Find the first time period where $i_\tau < 0$. Suppose $i_{t+1} < 0$ under the shock ε_t. Then, we want the following system to apply at time period $t+1$:

$$\begin{aligned}y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t\\ i_t &= 0,\end{aligned}$$

and the non-ZLB system to apply for t and time periods $\tau > t+1$. The system at $t+1$ translates into the following structural matrices:

$$\Gamma^*_{0,t+1} = \begin{bmatrix} 1 & 1 & -1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad \Gamma^*_{1,t+1} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\quad C^*_{t+1} = \begin{bmatrix}\bar{i}\\ 0\\ 0\end{bmatrix},\quad \Psi^*_{t+1} = \begin{bmatrix} 1\\ 0\\ 0\end{bmatrix},$$

while $\Pi^*_{t+1} = \Pi$. We now put these structural matrices in place in the format of (44):

$$\begin{bmatrix}\widetilde{\Gamma}_0 & 0\\ -\Gamma^*_{1,t+1} & \Gamma^*_{0,t+1}\\ 0 & \widetilde{Z}_2'\end{bmatrix}\begin{bmatrix}\mathbf{y}_t\\ \mathbb{E}_t\mathbf{y}_{t+1}\end{bmatrix} = \begin{bmatrix}\widetilde{C} + \widetilde{\Gamma}_1\mathbf{y}_{t-1} + \widetilde{\Psi}\varepsilon_t\\ C^*_{t+1}\\ \widetilde{w}_2\end{bmatrix}, \tag{48}$$

where $\widetilde{Z}_2'$ and $\widetilde{w}_2$ are the matrices defined in section 2.2 associated with the solution to the non-ZLB system. Invert (48) to obtain $\mathbf{y}_t$ and $\mathbf{y}_{t+1}$. Then use the solution to the non-ZLB system to obtain $\mathbf{y}_{t+j}$ for $j > t+1$ again assuming no future shocks.

   Return to step 2 with the the new path of $\{i_\tau\}_{\tau=t}^T$.

4. Examine the new path of $\{i_\tau\}_{\tau=t}^T$. If $i_\tau > 0\ \forall\ \tau$, then stop the algorithm: the ZLB applies only for time period $t+1$. Otherwise, move to the next step having already imposed the ZLB at time period $t+1$.

5. Find the new first time period where $i_\tau < 0$. Suppose $i_t < 0$ under the shock ε_t with $i_{t+1} = 0$. This could happen as the imposition of $i_{t+1} = 0$ is a contactionary monetary policy relative to $i_{t+1} < 0$. Then, we want the following system to apply at t:

$$\begin{aligned}y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t\\ i_t &= 0,\end{aligned}$$

   and the non-ZLB system to apply for time periods $\tau > t+1$. The system at t translates into the following structural matrices:

$$\widetilde{\Gamma}_0^* = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \widetilde{\Gamma}_1^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{C}^* = \begin{bmatrix} \bar{i} \\ 0 \end{bmatrix}, \quad \widetilde{\Psi}_t^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

Again, putting these structural matrices in place in the format of (44):

$$\begin{bmatrix} \widetilde{\Gamma}_0^* & 0 \\ -\Gamma_{1,t+1}^* & \Gamma_{0,t+1}^* \\ 0 & \widetilde{Z}_2' \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \mathbb{E}_t \mathbf{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \widetilde{C}^* + \widetilde{\Gamma}_1^* \mathbf{y}_{t-1} + \widetilde{\Psi} \varepsilon_t \\ C_{t+1}^* \\ \widetilde{w}_2 \end{bmatrix}.$$

And so, invert the LHS matrix to obtain the path of the model variables during the ZLB episode including the path of the nominal interest rate. Now the ZLB applies for two periods.

6. Continue iterating until the nominal interest rate satisfies the ZLB across the forecast horizon.

5 Result 1

Result 1. *If there are n distinct equations for n variables in the linearized model (**??**) under the non-ZLB regime and the Blanchard-Kahn conditions are satisfied for the linearized model under the non-ZLB regime, then the path during the ZLB period exists and is unique.*

Given the discussion in section 2.3, Result 2 involves checking the rank of the A matrix of equation (44). In particular, we need that the rank of A is $n \times T$. Without loss of generality, take the case where $T = 3$:

$$A = \begin{bmatrix} \widetilde{\Gamma}_{0,1} & 0 & 0 \\ -\Gamma_{1,2} & \Gamma_{0,2} & 0 \\ 0 & -\Gamma_{1,3} & \Gamma_{0,3} \\ 0 & 0 & \bar{Z}_2' \end{bmatrix}.$$

Uniqueness of the final solution implies the rank of $\bar{Z}_2' = k$. From our assumption that there are $n_1 + n_2$ unique equations of the original system, appending k expectations revisions to the system ensures there are $n = n_1 + n_2 + k$ unique equations to the non-ZLB system. And so, if all that has changed is that the interest rate rule is now specified to force the nominal interest rate to the zero, then we have:

$$\text{rank}\left(\begin{bmatrix} -\Gamma_{1,2} & \Gamma_{0,2} & 0 \\ 0 & -\Gamma_{1,3} & \Gamma_{0,3} \end{bmatrix} \right) = 2n.$$

It remains to argue that the final set of equations $\begin{bmatrix} 0 & 0 & \bar{Z}_2' \end{bmatrix}$ cannot contain an equation which is a linear combination of the equations in $\begin{bmatrix} 0 & -\Gamma_{1,3} & \Gamma_{0,3} \end{bmatrix}$. To see that this is the case, suppose that indeed there is an equation in $\begin{bmatrix} 0 & 0 & \bar{Z}_2' \end{bmatrix}$ that is a linear combination of the equations in $\begin{bmatrix} 0 & -\Gamma_{1,3} & \Gamma_{0,3} \end{bmatrix}$. Then we have that $\mathbb{E}_3 \mathbf{y}_{2,4}$ is defined in terms of $\mathbf{y}_{1,3}$ and $\mathbf{y}_{2,3}$ in the same way as implied by the system under the ZLB regime. In our application of the ZLB, this implies that there are some expectations that behave the same way as the structure when the interest rate is held constant. But this is not consistent with the final system implying a unique and determinate solution. Therefore, $\text{rank}(A) = 3 \times T$, as needed for A to be invertible.

6 Illustration of algorithm with the three equation New Keynesian model

The example in this section illustrates the algorithm and the identification method in a simulation of a simple model where known forward guidance policies are specified.

6.1 The model

The log-linearized economy is summarized by the following three equations, where i_t is the nominal interest rate, y_t is detrended output and π_t is the rate of inflation. There are three autoregressive shocks, to permanent technology z_t, to demand ξ_t and to the pricing equation a_t. There is a monetary policy shock $\varepsilon_{i,t}$ when the interest rate is positive. All variables are expressed as deviations from their steady-state values. The equations are, first, the Euler equation:

$$y_t = \mathbb{E}_t[y_{t+1}] - (i_t - \mathbb{E}_t[\pi_{t+1}]) + (1 - \rho_\xi)\xi_t, \tag{49}$$

second, the pricing equation:

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa[y_t - a_t], \tag{50}$$

and third, the policy rule, subject to the ZLB:

$$i_t = \max\left\{-i_{\text{ss}},\ \rho_i i_{t-1} + \phi_\pi \pi_t + \phi_g(y_t - y_{t-1} + z_t) + \varepsilon_{i,t}\right\}. \tag{51}$$

Since i_t is expressed as a deviation from steady-state, the ZLB binds when $i_t = -i_{\text{ss}}$ where $i_{\text{ss}} = \frac{\pi z}{\beta}$, with π^* and z^* being the inflation target and steady-state permanent rate of technology growth. The markup shock is autoregressive:

$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t},$$

as is the shock to technology:

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t},$$

and the demand shock:

$$\xi_t = \rho_\xi \xi_{t-1} + \varepsilon_{\xi,t}.$$

6.2 Simulation

The model is calibrated to values that reflect the estimated parameters in analogous New Keynesian models (as in, for example, Ireland, 2004). The following table gives the baseline calibration.

β	z	π	κ	ρ_i	ϕ_π	ϕ_g	ρ_a	ρ_z	ρ_ξ	σ_ξ	σ_a	σ_i	σ_z
.99	1.0025	$1.01^{1/4}$	.2	.8	1.7	.1	.8	.2	.8	.04	.01	.003	.01

The inflation target is set to 1 per cent to lower the steady-state nominal interest rate and make the ZLB more likely to be visited. Demand and pricing shocks are relatively more persistent than technology shocks and demand shocks are large compared to the other disturbances. Monetary policy reacts stronger to deviations in inflation from target as compared to the growth rate of output.

Figure 1 plots a simulated series of the interest rate, inflation and output generated by the model and a random set of shocks. The figure plots three series. The series labelled 'endogenous durations' is the series that transpires under the ZLB algorithm if the nominal interest rate was simply constrained by the ZLB, and becomes positive as soon as the policy rule requires it. The series labelled 'no ZLB' is the simulated series where the ZLB is unconstrained. I plot the unconstrained path as a comparison to the third series labelled 'forward guidance', which is the simulated path where the central bank announces and commits to a path for the interest rate at zero for a period of time. The path the central bank commits to, as summarized by a sequence of anticipated durations, is illustrated in Figure 2 and discussed below.

This particular set of shocks drives the nominal interest rate to the ZLB in period 13 and keeps it there until period 16. In period 17, the shocks lead the central bank to raise the interest rate, but it hovers around zero until period 22. Under the forward guidance path, the interest rate stays at zero until period 19 and returns to the ZLB for a single period at period 21. The decline in output and inflation is much more pronounced in the endogenous durations path as compared to the path under active forward guidance.

The durations that agents in the model expect the ZLB to bind at each period are plotted in Figure 2. The first panel shows the anticipated durations for the endogenous durations case. The shocks keep the anticipated duration at 2 periods in both periods 14 and 15. This stability in the anticipated duration corresponds to a decline in inflation and output from period 14 to period 15, so that this particular sequence of shocks pushes out the ex-post ZLB exit date by one period.

The second panel of Figure 2 shows the anticipated durations under the forward guidance policy. The central bank announces that it will abandon its policy rule for the specified duration and credibly commit to holding the nominal interest rate at zero for that duration. This announcement could, in practice, take any value. However, if the announced path is less than or the same as the endogenous duration, the announcement has no aggregate effects because the announcement confirms agents' expectations of the ZLB duration. Furthermore, to ensure an announcement policy is time-consistent, if the central bank commits to a particular duration T, then in the next time period I require it to commit to a duration which is at least as high as $T - 1$. While in some sense arbitrary, the forward guidance values chosen for this exercise can be rationalized with a policy rule which activates when the interest rate becomes zero and which is tied to the rate of inflation that would arise in the absence of forward guidance policies. An example rule with this feature is outlined and studied in section **??**.

6.3 Illustrating the decomposition

The aggregate consequences of forward guidance announcements, as plotted in Figure 1, illustrates how the anticipated duration series is a crucial variable when the interest rate is at zero. Under the forward guidance path, both output and inflation lie well above the paths that arise when no forward guidance is used. Information on the ZLB durations together with the observable variables can be submitted to the ZLB algorithm to identify how stimulatory those announcements are in practice.

To illustrate this, assume we observe the interest rate, inflation and output series under the 'forward guidance' series, and we estimate the anticipated ZLB durations under these series. Suppose the procedure perfectly estimates the parameters and the ZLB durations, so that they exactly equal the durations shown in the second panel of Figure 2. The smoother provides an estimate of the structural shocks $\{w_\tau\}_{\tau=1}^{T}$. Using the estimated structural shocks, the model and the ZLB algorithm, the third panel of Figure 2 illustrates the estimated ZLB durations as decomposed into the component due to structural shocks and the component due to active forward guidance policy. The identification procedure finds that forward guidance contributed an additional two periods to each duration from time periods 13 to 17, and an additional period in time period 18.

Figure 3 illustrates the identification procedure period-by-period. The first time period the ZLB binds is in period 13, plotted in the first panel. At this period, the estimated duration is five periods. Also plotted is the forecast of the interest rate that is implied by the state x_{t-1} and structural shock w_t at time period 13, labelled the 'shadow rate'. The duration is at zero for two periods longer than the shadow rate is at zero, so that the identified forward guidance component is two periods. Most strikingly, in time period 17, the shadow rate is positive for the two periods that the ZLB is estimated to bind.

The example simulation makes it clear that the identified endogenous duration is not simply the difference between durations that arise when there is no forward guidance and the durations that arise under forward guidance (or the first two panels of Figure 2). This is because each announced duration under forward guidance changes the state vector at that period, and the different path of the state vector could change the endogenous binding of the ZLB. This is why, in the simulation, the ZLB also binds for one period endogenously in period 21–because the interest rate was kept at zero for longer under the forward guidance simulation, the desire of the central bank to smooth interest rates keeps it closer to zero relative to the no forward guidance simulation, so that a deflationary shock is more likely to make the ZLB bind after $t = 18$.

7 Comparison to non-linear approximation

To show that the method provides a good approximation, here I compare the output of the algorithm to a non-linear approximation of a simplified version of the Ireland (2004) economy. The non-linear model solved consists of an equation relating consumption c_t to output y_t and includes a term for price adjustments costs:

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t,$$

where π_t is the inflation rate. The Euler equation is:

$$\frac{1}{\beta}\frac{1}{i_t}\frac{a_t}{c_t} = \mathbb{E}_t\left[\frac{a_{t+1}}{c_{t+1}}\frac{1}{\pi_{t+1}}\right],$$

where i_t is the gross nominal interest rate, a_t is a demand shock, and z_t is a permanent productivity shock, and an equation derived from intermediate goods producing firms optimal price adjustment:

$$\mathbb{E}_t\left[\frac{a_{t+1}}{a_t}\frac{y_{t+1}}{c_{t+1}}\left(\frac{\pi_{t+1}}{\pi}-1\right)\frac{\pi_{t+1}}{\pi}\right]=\frac{1}{\beta\phi}\frac{y_t}{c_t}\left[\theta_t-1-\theta_t\left(\frac{c_t}{a_t}\right)y_t^{\eta-1}+\phi\left(\frac{\pi_t}{\pi}-1\right)\frac{\pi_t}{\pi}\right],$$

where θ_t is a shock to intermediate goods producing firms' desired markup. The central bank follows a Taylor rule. The ZLB on the gross nominal interest rate requires $i_t > 1$, so the rule becomes:

$$i_t = \max\left[1, \pi_t^{\rho_\pi} g_t^{\rho_g} x_t^{\rho_x}\right],$$

where g_t is the growth rate of output from $t-1$ to t:

$$g_t = \frac{y_t}{y_{t-1}} z_t,$$

and x_t is the efficient level of output:

$$x_t = \frac{y_t}{a_t^{1/\eta}}.$$

The demand shocks follows an autoregressive process:

$$\ln(a_t) = (1-\rho_a)\ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{a,t}.$$

The two state variables of the model are a_{t-1} and y_{t-1}. To approximate the solution, I follow the exposition of Fernández-Villaverde et al. (2012). I first discretize a_t with a Tauchen approximation. The solution will be in terms of two functions $f_1(y_{t-1}, a_t)$ and $f_2(y_{t-1}, a_t)$ which approximate the expectations given by $\frac{1}{\beta}\frac{1}{i_t}\frac{a_t}{c_t}$ and $\frac{a_t}{\beta\phi}\frac{y_t}{c_t}\left[\theta-1-\theta\left(\frac{c_t}{a_t}\right)y_t^{\eta-1}+\phi\left(\frac{\pi_t}{\pi}-1\right)\frac{\pi_t}{\pi}\right]$ respectively. First, I constrain y_{t-1} to lie between 0.95 and 1.05 times the steady-state value of y. Together with the discretized a_t, this gives a grid $\{\mathbf{y}, \mathbf{a}\}$. I use the guess-and-verify method to approximate the policy functions across that grid. The algorithm is:

1. Guess the values of $f_1(s^i)$ and $f_2(s^i)$ at each point $s^i \in \{\mathbf{y}, \mathbf{a}\}$. Call these guesses $\hat{f}_1(s^i)$ and $\hat{f}_2(s^i)$.

2. Using the guesses $\hat{f}_1(s^i)$ and $\hat{f}_2(s^i)$:

   for each i (for each state):

   Guess π_t . Using the guess for π_t:

   (a) Determine the ratio $\frac{c_t}{y_t}$.

   (b) Using the ratio $\frac{c_t}{y_t}$ and the guess $\hat{f}_2(s^i)$, obtain y_t.

   (c) Using y_t, obtain c_t, a_t, g_t and then i_t, where i_t is calculated subject to $i_t = 1$.

   (d) Using $\hat{f}_1(s^i)$ and i_t, compute the implied consumption and call it $\widetilde{c}_t$.

   Check the computed $\widetilde{c}_t$ against c_t. Update guess of π_t until $|\widetilde{c}_t - c_t|$ converges.

3. With the equilibrium policy functions at time t, compute the expectations:

$$\mathbb{E}_t\left[\frac{a_{t+1}}{c_{t+1}}\frac{1}{\pi_{t+1}}\right] \quad \text{and} \quad \mathbb{E}_t\left[a_{t+1}\frac{y_{t+1}}{c_{t+1}}\left(\frac{\pi_{t+1}}{\pi}-1\right)\frac{\pi_{t+1}}{\pi}\right],$$

using the transition matrix for the discretized θ_t. Adjust the guesses $\hat{f}_1(s^i)$ and $\hat{f}_2(s^i)$ until they converge with the computed $f_1(s^i)$ and $f_2(s^i)$.

The algorithm approximates policy functions for all the endogenous variables.

I use the following calibration, which mirrors the calibrated and estimated results of Ireland (2004). The inflation target is set to 1 per cent per year so that the ZLB is more likely to bind following a negative demand shock.

β	ϕ	η	ρ_π	ρ_g	ρ_x	ρ_a	σ_a	θ	π
0.99	200	1/0.06	2.5	0.3	0.1	0.7	0.02	$0.1\frac{\phi}{\eta}+1$	$1.01^{1/4}$

Figure 4 compares the impulse response to a large negative five standard deviation demand shock under the non-linear approximation and the ZLB algorithm. Both methods have similar profiles for output growth and inflation, and show that the ZLB binds for two periods following the shock.

Figure 5 shows the two approximations have similar predictions for the stochastic paths of endogenous variables under 1000 simulations using the same set of shocks for both approximations.

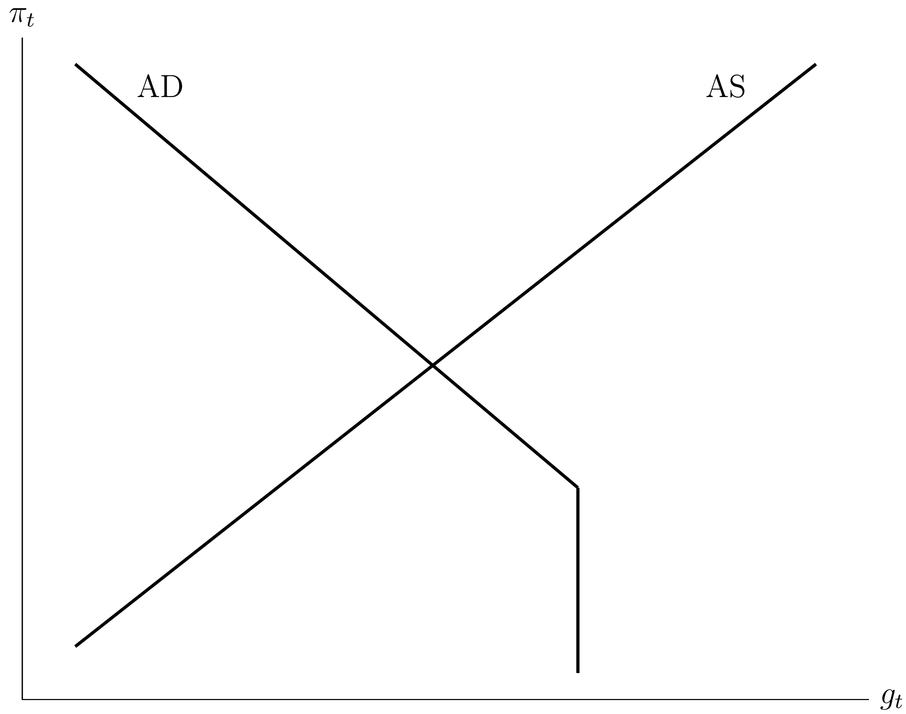
7.1 Severity of shocks at the ZLB

I use the implementation to analyse the severity of shocks when the nominal interest rate is at the ZLB. Figure 1 showed that, when the central bank cannot stabilize a large negative demand shock pushing the nominal interest rate to its lower bound, the inflation and output decline is pronounced. I investigate this further in Figure 6 by plotting the initial response of inflation following demand (ξ_t) shocks of varying size. The non-stochastic response is the bold black line. Also plotted are fancharts for the range of the initial response when the economy is subject to random shocks to the other stochastic variables.

For the region in which the ZLB does not bind, the initial response is linear. Beyond large, negative demand shocks, the ZLB binds and the decline in inflation and output is more severe with more negative shocks. The width of the fanchart also widens for both inflation and output when the ZLB binds. To make this clear, Figure 7 plots a normalized measure of the width of the bands around the initial response in Figure 6. This exercise illustrates how, when the central bank is constrained and unable to act against the shock with the nominal interest rate, the effect of unanticipated stochastic shocks is large. This is intuitive: policy functions for inflation and output are steeper at the ZLB relative to when the central bank can stabilize contractionary demand shocks, and so further shocks that impact when the nominal rate is at that bound moves the economy along those steeper functions.

To illustrate the severity of shocks which impact the economy at the ZLB over time, Figure 8 plots fancharts summarizing the range of the interest rate, inflation and output growth across 100 simulations of the model when the interest rate is subject to the ZLB, and for when it is not subject to the ZLB. The shocks are such that the economy is subject to three consecutive quarters of unanticipated negative two standard deviation risk premia

AD/AS under ZLB



shocks, in addition to random shocks. This pushes the nominal interest rate to its ZLB for an extended period. The black line gives the economy's path under three consecutive quarters of negative two standard deviation risk premia shocks and absent further stochastic shocks. The fancharts illustrate how, when the ZLB binds, some paths are particularly variable, so that the overall variance of inflation and output growth rises over the simulation period when compared to the variance in the no-ZLB simulation.

The figure below illustrates the aggregate demand (AD) and aggregate supply (AS) curves in the inflation-output growth space under the ZLB,[2] and motivates why shocks can increase the volatility of inflation and output growth, as a given shock affects inflation and output growth differently depending on whether the equilibrium lies on the sloped or vertical segment of the AD curve. The vertical component of the AD curve arises because monetary policy cannot manipulate the real interest rate when it is constrained by the ZLB. A given shift in the AS curve along the vertical component of AD can cause inflation to be more volatile, while a given shift in the AD curve when the equilibrium lies in the vertical component of the AD curve can cause both inflation and output growth to be more volatile.

[2]See the next section for the derivation of these curves in the three-equation New Keynesian model. If derived in the space of expected inflation $\mathbb{E}_t\pi_{t+1}$ and output as in Wieland (2014) the AD curve would be upward sloping in the ZLB region.

8 Aggregate demand and supply under ZLB

In this section, I derive the three equation New Keynesian model in the AD and AS framework as in Jones and Kulish (2016) but with the ZLB. Define output growth:

$$g_t = y_t - y_{t-1} + z_t. \tag{52}$$

To find the AS schedule, substitute equation (52) into (50) to get:

$$\pi_t = \kappa g_t + s_t + (\pi - \kappa g), \tag{53}$$

where $s_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_{t-1} - \kappa z_t - \kappa a_t$. In the space of contemporanous output growth and inflation (g_t, π_t), equation (53) expresses inflation as a linear function of output growth, with slope κ and intercept $s_t + (\pi - \kappa g)$. Note that the time-varying component of the intercept s_t is zero when the economy is on its growth path. Also note that the slope of the curve depends on the degree of nominal price rigidities. That is, as $\kappa \to \infty$ prices become fully flexible implying a vertical AS. Conversely as the cost of price adjustment rises, $\kappa \to 0$, so that AS flattens.

To obtain the AD schedule when the ZLB does not bind, substitute equations (51) and (52) into (49) to get:

$$\pi_t = -\left(\tfrac{1+\psi_g}{\psi_\pi}\right) g_t + d_t + \left(\pi + \left(\tfrac{1+\psi_g}{\psi_\pi}\right) g\right), \tag{54}$$

where $\psi_\pi d_t = -\rho_i i_{t-1} + \mathbb{E}_t y_{t+1} + \mathbb{E}_t \pi_{t+1} - y_{t-1} + z_t + (1-\rho_\xi)\xi_t - \varepsilon_{i,t}$. Note that, as for the time-varying intercept in the AS curve, when the economy is on its balanced growth path, d_t is zero. The slope of the curve (54) depends on the parameters of the policy rule. A greater response to deviations of inflation from target, ψ_π, flattens the curve. Vice versa, stronger responses to output growth, ψ_g, steepen the AD curve. The central bank uses changes in the nominal interest rate to affect growth and stabilize the inflation rate around the target.

The AD and AS curves reveal that shocks move both schedules simultaneously and so, to determine the overall effect of the shocks on π_t and g_t, we find their intersection. With the values of s_t and d_t in hand, the AS curve (53) and the AD curve (54) can be written as a system of two equations in two variables g_t and π_t. Inverting these equations:

$$\begin{bmatrix} \pi_t \\ g_t \end{bmatrix} = \begin{bmatrix} \pi + \frac{\kappa\psi_\pi}{1+\psi_g+\kappa\psi_\pi} d_t + \frac{1+\psi_g}{1+\psi_g+\kappa\psi_\pi} s_t \\ g + \frac{\psi_\pi}{1+\psi_g+\kappa\psi_\pi} d_t - \frac{\psi_\pi}{1+\psi_g+\kappa\psi_\pi} s_t \end{bmatrix}. \tag{55}$$

Now suppose the ZLB binds. The AD curve changes, while the AS curve is unchanged. To derive the new AD curve, notice that the rule (51) says that, at the ZLB, the parameters of the rule are equal to zero. This says the AD curve becomes vertical at a level of g_t determined by substituting (51) with the constraint binding, and (52) into (49) to get:

$$g_t = \widetilde{d}_t + g,$$

where $\widetilde{d}_t = -i_{\text{ss}} + \mathbb{E}_t y_{t+1} + \mathbb{E}_t \pi_{t+1} - y_{t-1} + z_t + (1-\rho_\xi)\xi_t$. The vertical component of the AD curve says that when the ZLB binds, the central bank cannot engineer an expansion of output by lowering the nominal interest rate.

To determine inflation and output growth at an equilibrium where the ZLB binds, write the AD and AS equations in inflation and output growth space, and solve:

$$\begin{bmatrix} \pi_t \\ g_t \end{bmatrix} = \begin{bmatrix} \pi + s_t + \kappa \widetilde{d}_t \\ g + \widetilde{d}_t \end{bmatrix}. \tag{56}$$

Comparing (55) with (56) reveals that shocks to e_t, a_t and z_t move both AD and AS simultaneously, directly through the shock and indirectly through expected inflation and output.

References

Binder, M. and H.M. Pesaran, "Multivariate Rational Expectations Models and Macroeconomic Modelling: A Review and Some New Results," Cambridge Working Papers in Economics 9415, Faculty of Economics, University of Cambridge 1995.

Cagliarini, Adam and Mariano Kulish, "Solving Linear Rational Expectations Models with Predictable Structural Changes," *Review of Economics and Statistics*, March 2013, *95* (1), 328–336.

Fernández-Villaverde, Jesús, Grey Gordon, Pablo A. Guerrón-Quintana, and Juan Rubio-Ramírez, "Nonlinear Adventures at the Zero Lower Bound," working paper 18058, NBER 2012.

Hamilton, James D., *Time Series Analysis*, Princeton University Press, 1994.

Ireland, Peter, "Technology Shocks in the New Keynesian Model," *Review of Economics and Statistics*, November 2004, *86* (4), 923–936.

Jones, Callum and Mariano Kulish, "A Graphical Representation of an Estimated DSGE Model," *Applied Economics*, 2016, *48* (6), 483–489.

Kulish, Mariano and Adrian Pagan, "Estimation and Solution of Models with Expectations and Structural Changes," *Journal of Applied Econometrics*, 2017.

Sims, Christopher, "Solving Linear Rational Expectations Models," *Computational Economics*, 2002, *20*, 1–20.

Smets, Frank and Rafael Wouters, "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, 2007, *96* (3), 586–606.

Wieland, Johannes, "Are Negative Supply Shocks Expansionary at the Zero Lower Bound?," Manuscript January 2014.

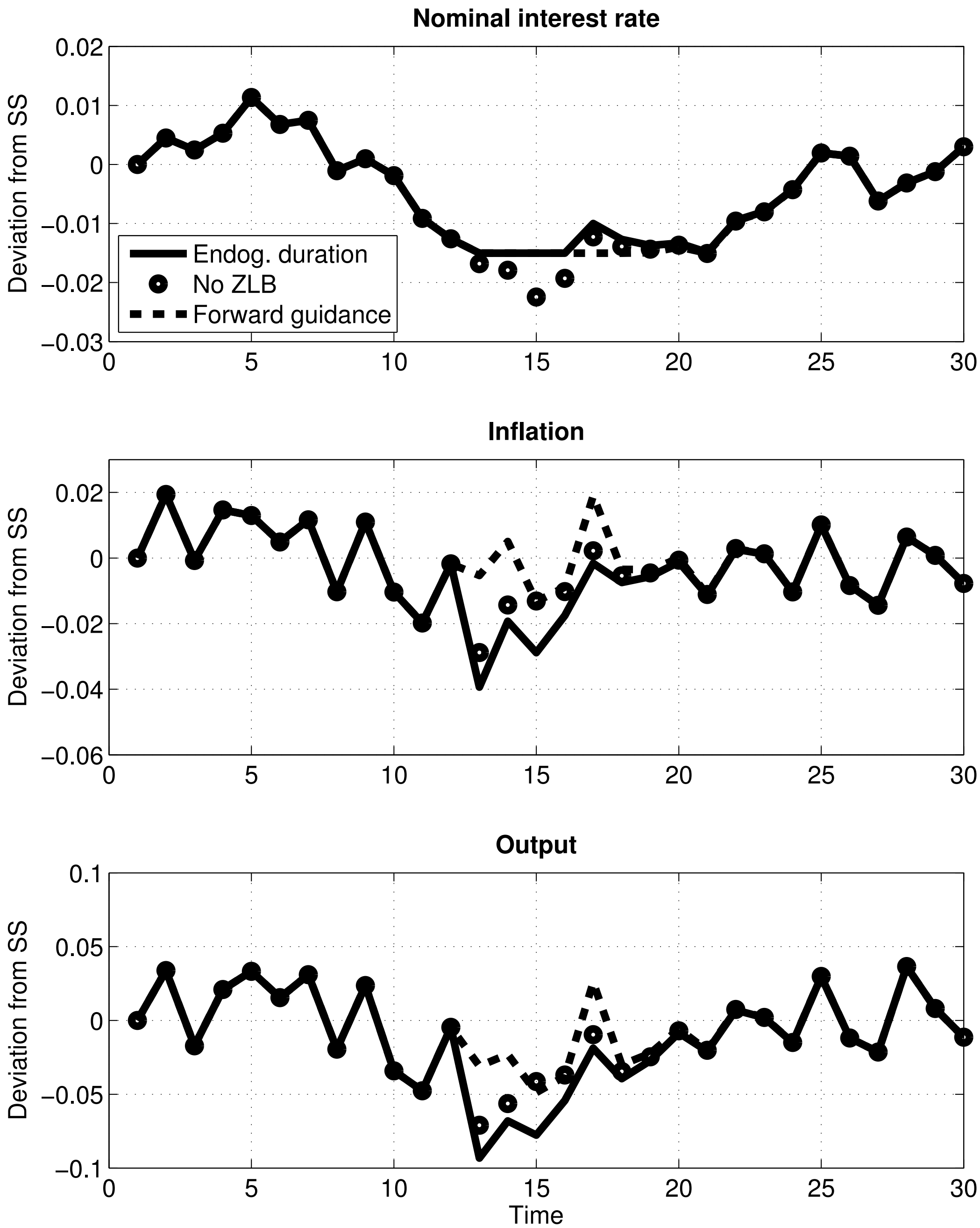


Figure 1: Simulation. This figure shows the interest rate, inflation, and output under a set of random shocks, when the interest rate is subject to the ZLB (without forward guidance), when there is no ZLB imposed, and when the central bank announces a sequence of interest rate ZLB durations above those implied endogenously by the shocks.

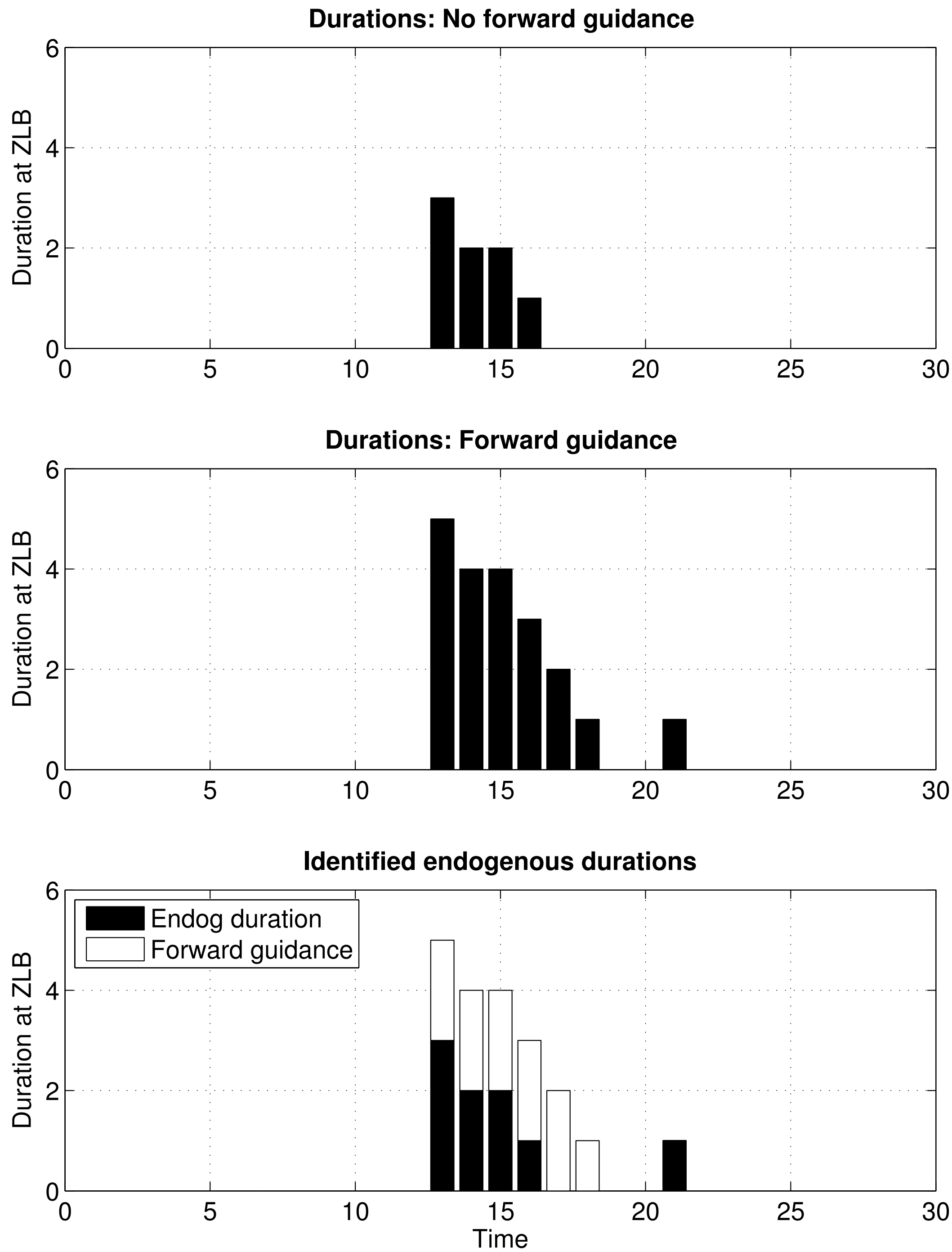


Figure 2: Estimated and decomposed durations. The top two panels of this figure show the anticipated durations of the ZLB in the 'endogenous durations' (no forward guidance) and 'forward guidance' simulations. The third panel shows the outcome of the identification procedure using the forward guidance durations.

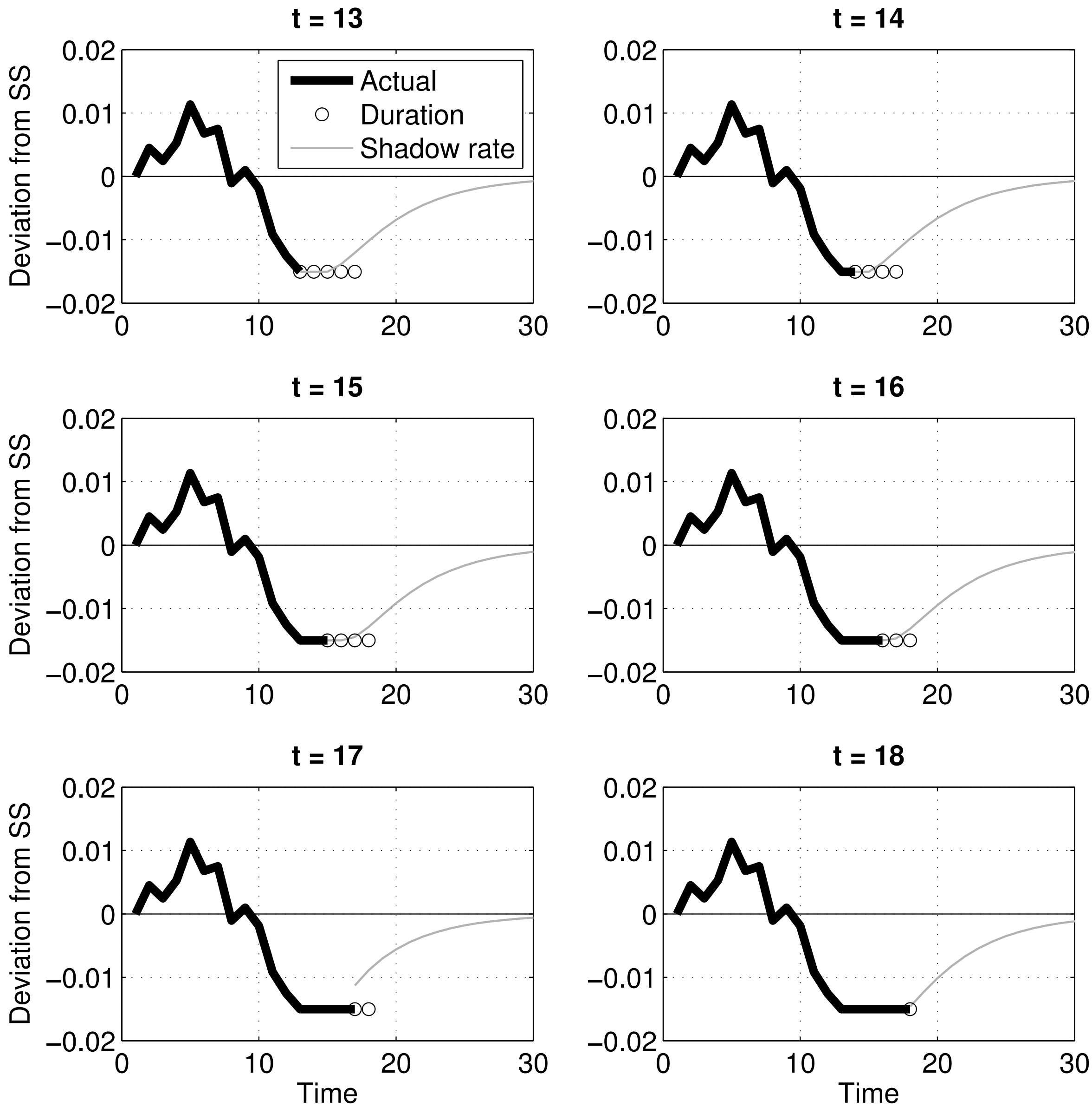


Figure 3: Interest rates during the ZLB episode. The figure shows, from time periods 13 to 18, forecasts of the interest rate under the estimated duration, and forecasts of the shadow interest rate. Forward guidance is active when the shadow rate forecast lies above the path under the estimated duration.

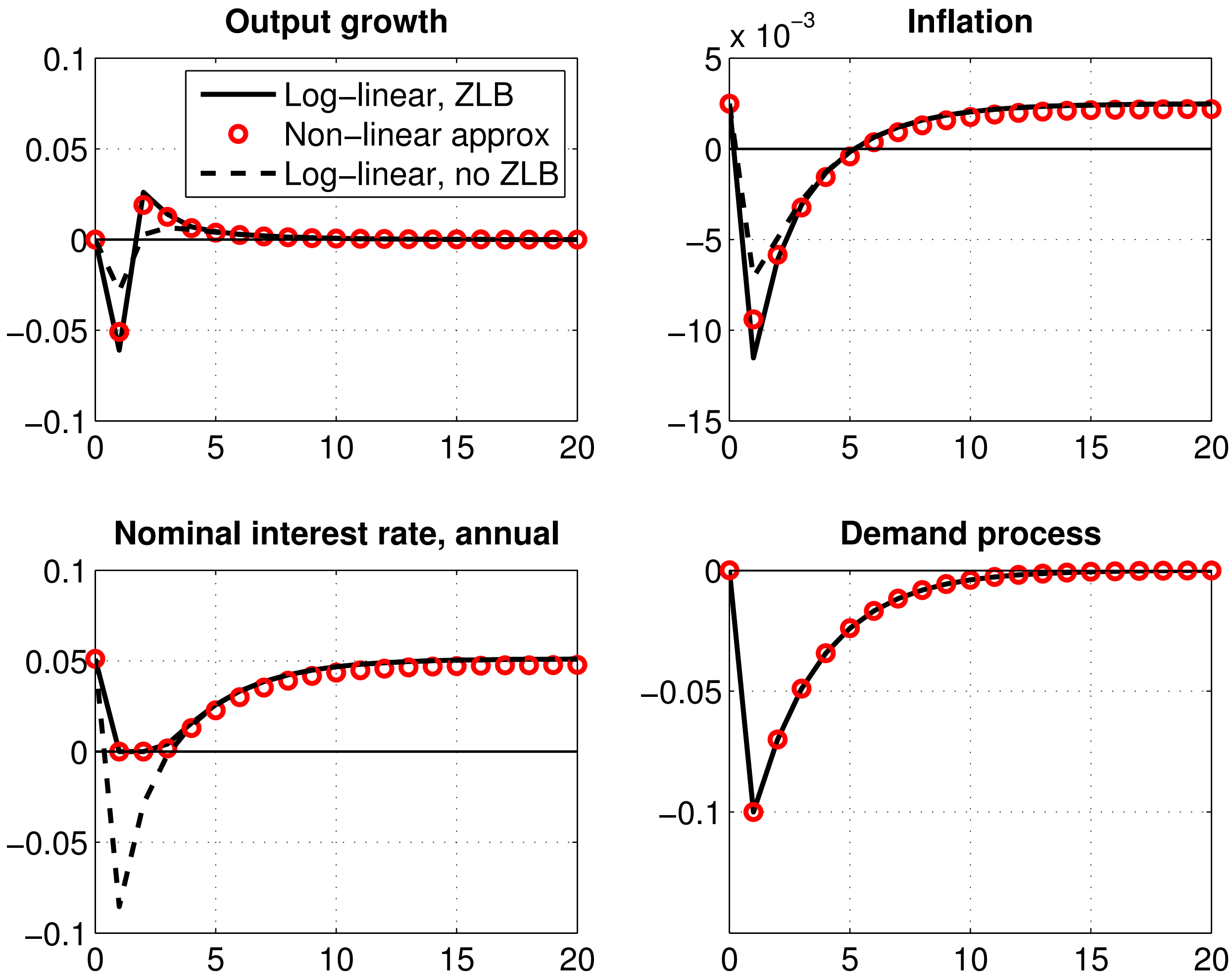


Figure 4: Comparison of ZLB algorithm and non-linear approximation. This figure plots the impulse response to a negative five standard deviation demand shock.

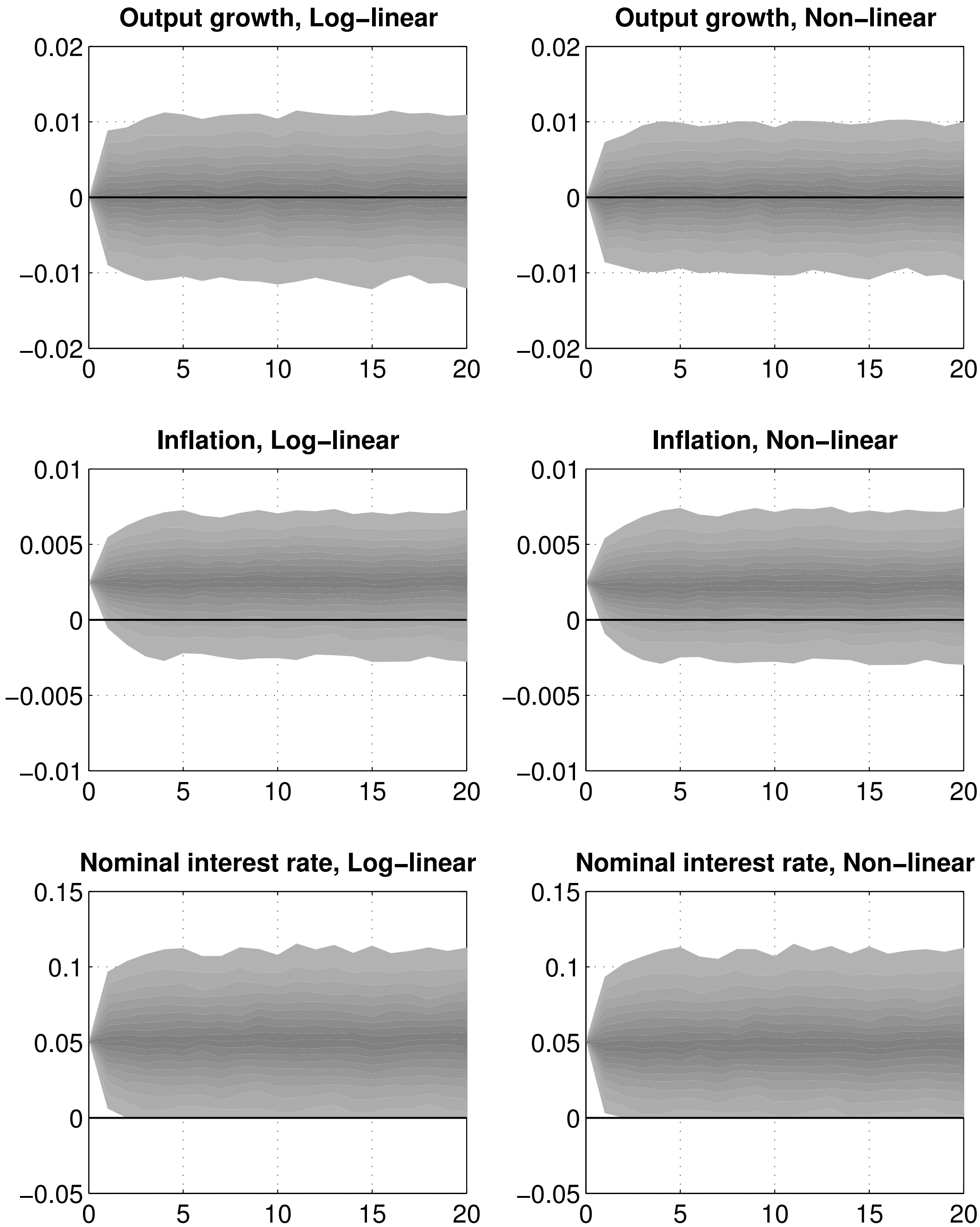


Figure 5: Comparison of ZLB algorithm and non-linear approximation over time. This figure plots fancharts for 1000 simulations of the log-linear and non-linear approximations under the same set of demand shock.

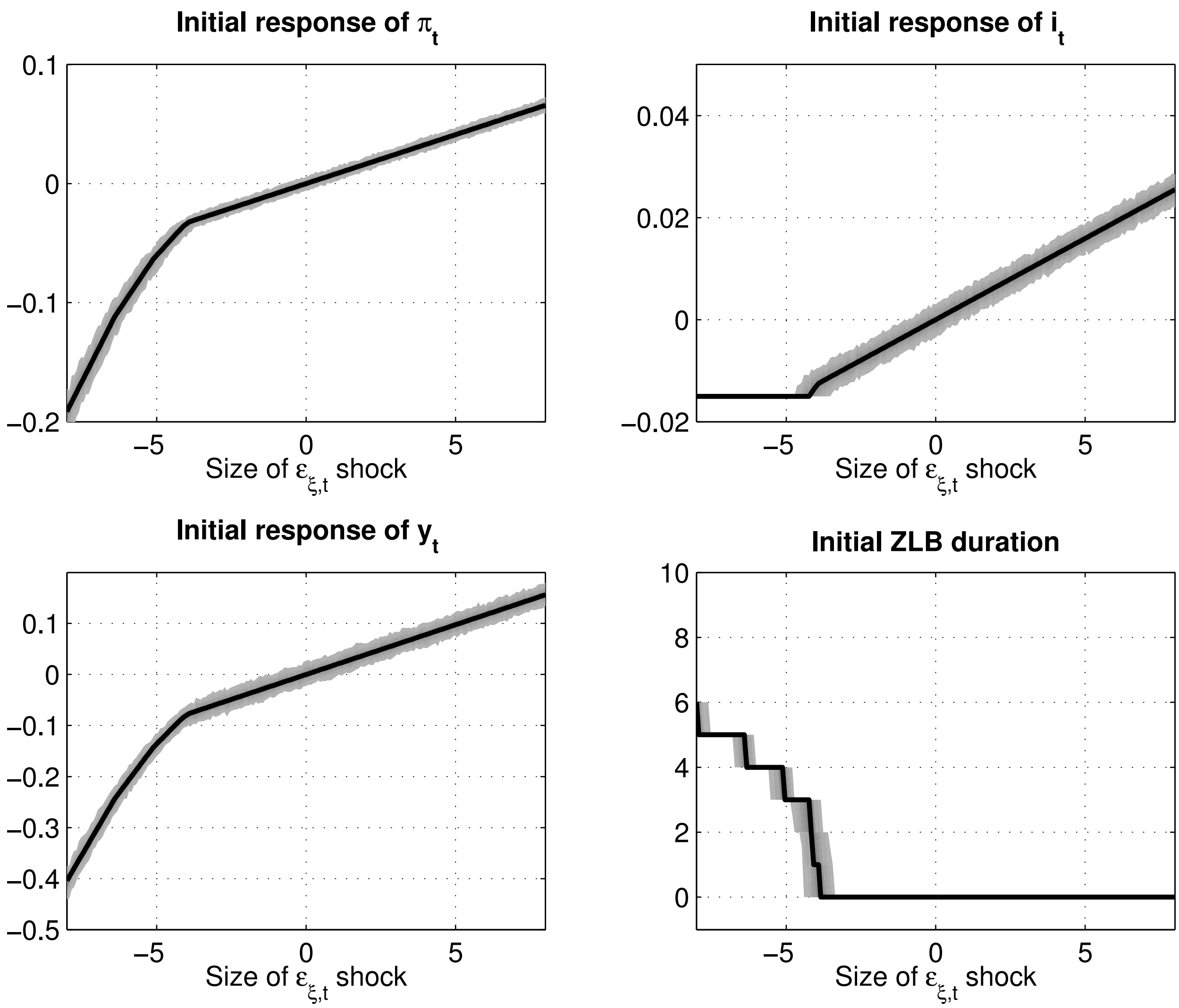


Figure 6: Initial response of variables to $\varepsilon_{\xi,t}$ shocks. This figure shows the initial responses of inflation, the nominal interest rate and output in response to demand shocks of varying size. The fanchart illustrates the band of the initial response when the economy is hit by other stochastic shocks.

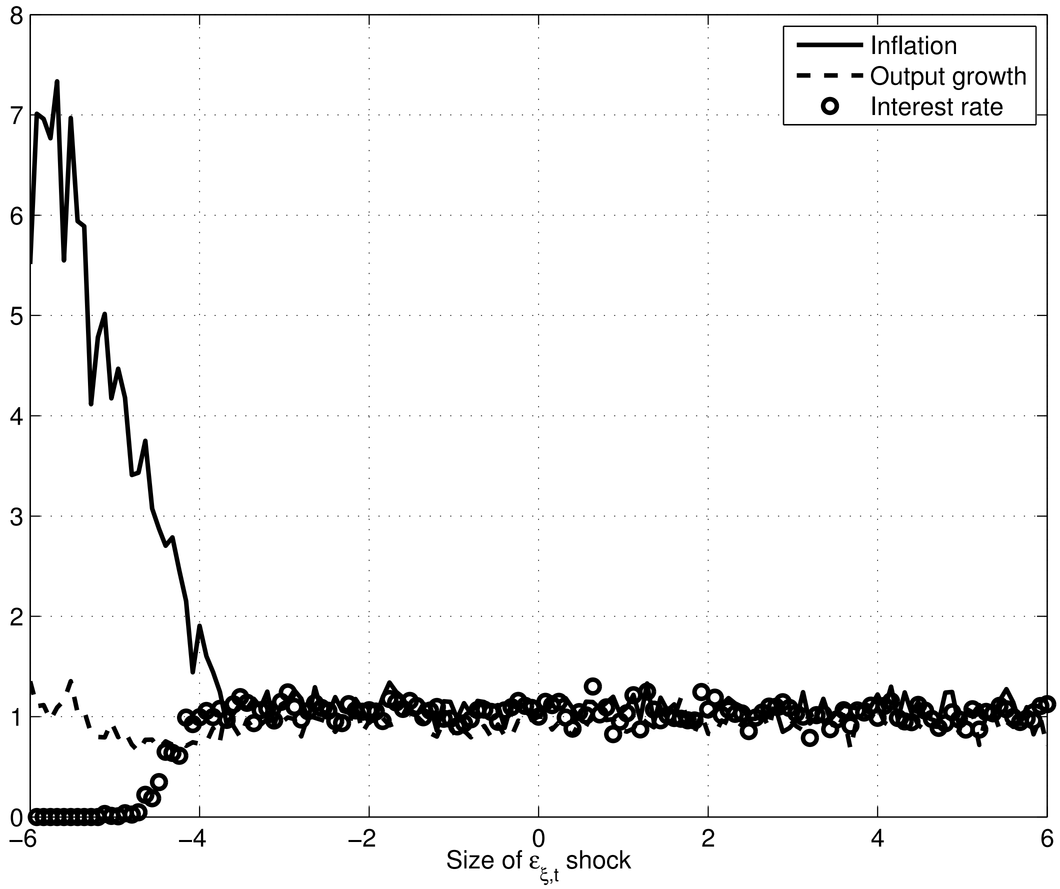


Figure 7: Standard Deviation of initial response of interest rate, inflation and output growth. The standard deviations of the interest rate, inflation and output growth are normalized by their respective standard deviations for $\varepsilon_\xi = 0$. The figure plots a normalized measure of the width of the fancharts around the initial responses of inflation, output growth and the interest rate, plotted in Figure 6.

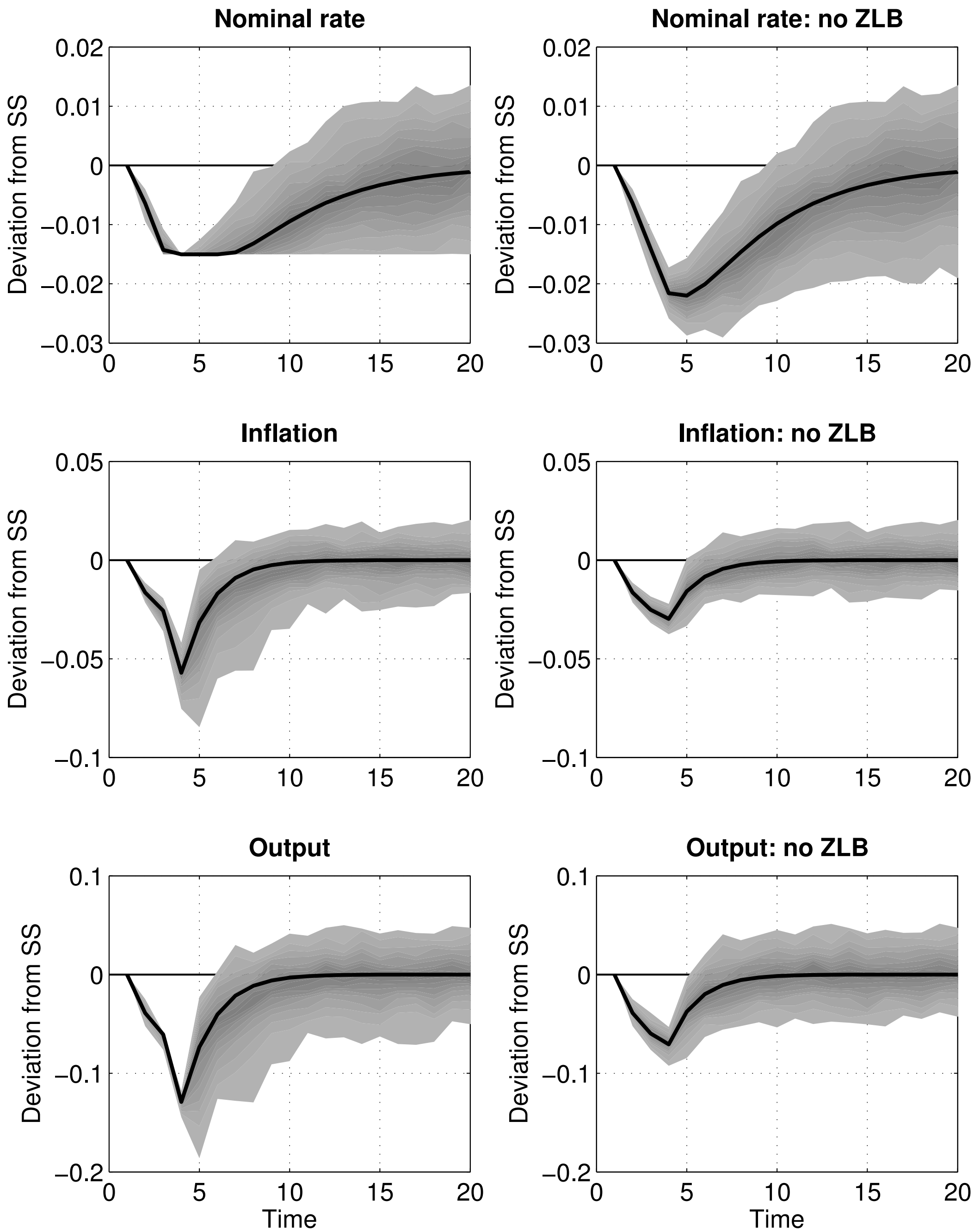


Figure 8: Unanticipated shocks each period. This figure plots fancharts of 100 simulations of the model economy when the ZLB binds and when it does not. Under the ZLB, the fancharts are wider, illustrating how shocks which hit at the ZLB can generate excess volatility.